QUANTUM MECHANICS IN PHASE SPACE: THE SCHRÖDINGER AND THE MOYAL REPRESENTATIONS

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ABSTRACT. We present a phase space formulation of quantum mechanics in the Schrödinger representation and derive the associated Weyl pseudo-differential calculus. We prove that the resulting theory is unitarily equivalent to the standard "configuration space" formulation and show that it allows for a uniform treatment of both pure and mixed quantum states. In the second part of the paper we determine the unitary transformation (and its infinitesimal generator) that maps the phase space Schrödinger representation into another (called Moyal) representation, where the wave function is the cross-Wigner function familiar from deformation quantization. Some features of this representation are studied, namely the associated pseudo-differential calculus and the main spectral and dynamical results. Finally, the relation with deformation quantization is discussed.

1. INTRODUCTION

A key principle of quantum mechanics states that the fundamental configuration and momentum operators satisfy the commutation relations of the Heisenberg algebra

$$(1.1) \qquad [\hat{x}_j, \hat{\xi}_k] = i\delta_{jk} \quad j, k = 1, .., n$$

all the other commutators being zero. The most standard implementation of this algebra is given by the Schrödinger representation, where $\hat{x}_j$ and $\hat{\xi}_j$ are viewed as self-adjoint operators

$$\hat{x}_j = \text{multiplication by } x_j \quad , \quad \hat{\xi}_j = -i\partial_{x_j}$$

acting on the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable functions (with support) on the classical configuration space $\mathbb{R}^n$. It is well known that the quantization rules

$$x_j \longrightarrow \hat{x}_j \quad , \quad \xi_j \longrightarrow \hat{\xi}_j$$

do not provide the complete information on how to quantize an arbitrary classical observable since the formal prescription $a(x, \xi_x) \longleftrightarrow \hat{a} = a(\hat{x}, \hat{\xi}_x)$ (where $x = (x_1, ..., x_n)$, $\xi_x = (\xi_1, ..., \xi_n)$) is order ambiguous. The Weyl pseudo-differential calculus yields the standard (but not unique) solution for this problem [31, 17, 32]. The Weyl correspondence $a \overset{\text{Weyl}}{\longleftrightarrow} \hat{a}^W$ is a one-to-one linear map that associates with each symbol $a \in S'(\mathbb{R}^{2n})$ a linear

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operator $\hat{a}^W : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, uniquely defined by

$$\hat{a}^W = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} F_\sigma a(x_0, \xi_0)\hat{T}(x_0, \xi_0) \, dx_0 \, d\xi_0$$

where the integral is a Bochner (operator) integral,

$$(1.2) \qquad F_\sigma a(x_0, \xi_0) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} a(x, \xi_x)e^{i(x_0 \cdot \xi_x - \xi_0 \cdot x)} \, dx \, d\xi_x$$

is the symplectic Fourier transform of a and $\hat{T}$ is the Heisenberg-Weyl operator, defined for all $\psi \in L^2(\mathbb{R}^n)$ by

$$T(x_0, \xi_0)\psi(x) = e^{i(\xi_0 \cdot x - \frac{1}{2}\xi_0 \cdot x_0)}\psi(x - x_0)$$

We now observe that:

(1) The Schrödinger representation is by no means the unique possible implementation of the commutation relations (1.1). The momentum representation in $L^2(\mathbb{R}^n)$ is a well known alternative. More generally, any unitary operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ generates another "quantization rule" through the prescription

$$x \longrightarrow U\hat{x}U^{-1} \qquad \xi_x \longrightarrow U\hat{\xi}_x U^{-1}$$

(2) Other interesting representations can be obtained by implementing the observables x and ξ_x as operators acting on Hilbert spaces, other than $L^2(\mathbb{R}^n)$.

This second possibility was seriously considered in a series of papers [29, 16, 18, 19, 20, 21, 9] focusing on representations in terms of operators acting on the Hilbert space $L^2(\mathbb{R}^{2n})$ of functions with support on the *phase space* $\mathbb{R}^{2n}$. The Frederick and Torres-Vega representation [29, 16],

$$x \longrightarrow \hat{x} = x + i\frac{\partial}{\partial p} \quad , \quad \xi_x \longrightarrow \hat{\xi}_x = -i\frac{\partial}{\partial x}$$

leading to the Schrödinger equation in the phase space

$$i\frac{\partial}{\partial t}\Psi(x, p, t) = H(x + i\frac{\partial}{\partial p}, -i\frac{\partial}{\partial x})\Psi(x, p, t)$$

and the "Moyal" representation (given by the "Bopp shifts" [8])

$$(1.3) \qquad x \longrightarrow \tilde{X} = x + \frac{1}{2}i\frac{\partial}{\partial p} \quad , \quad \xi_x \longrightarrow \tilde{\Xi}_x = p - \frac{1}{2}i\frac{\partial}{\partial x}$$

are two examples of this sort.

The latter, more symmetric, representation leads to what we shall call the "Moyal-Weyl pseudo-differential calculus" originally presented in [20] and further studied, in connection with the related "Landau-Weyl" calculus, in [18, 21]. The representation eq. (1.3) is intimately connected to the

deformation formulation of quantum mechanics. Indeed, for an arbitrary Weyl symbol $a \overset{\text{Weyl}}{\longleftrightarrow} \hat{a}^W$, we have in the Moyal representation [19, 20]

$$a(\tilde{X}, \tilde{\Xi}_x) = a(x, p) \star$$

where $\star$ is the Moyal starproduct [5, 32]. Hence, the stargenvalue equation

(1.4) $$a(x, p) \star \Psi_\lambda(x, p) = \lambda \Psi_\lambda(x, p)$$

can be written in the form

(1.5) $$a(\tilde{X}, \tilde{\Xi}_x) \Psi_\lambda(x, p) = \lambda \Psi_\lambda(x, p)$$

Moreover, it was also proved in [19, 20] that the solutions of the previous eigenvalue equation (1.5) are related to the solutions of the usual eigenvalue equation

(1.6) $$a(\hat{x}, \hat{\xi}_x) \psi_\lambda(x) = \lambda \psi_\lambda(x)$$

by the action of intertwining operators $W_\phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ defined for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\Psi(x, p) = W_\phi \psi(x, p) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-ip \cdot y} \psi(x + \frac{1}{2}y) \phi^*(x - \frac{1}{2}y) \, dy$$

and related to the cross Wigner distribution $W(\psi, \phi)$ by a simple normalization factor

$$W_\phi \psi = (2\pi)^{n/2} W(\psi, \phi).$$

Combining the two results (the equality of the two equations (1.4) and (1.5) and the relation - given by W_ϕ - of the solutions of eq.(1.5) with those of eq.(1.6)), de Gosson and Luef [20] found a simple proof of the spectral relation between Schrödinger quantum mechanics and the deformation quantization of Bayen *et al.* [5, 6].

Some of these results were generalized in [13] where an extension of the "phase space Moyal representation" (1.3) and the associated pseudo-differential calculus, allowed for the precise construction of the quantum theory associated with the extended Heisenberg algebra. The resulting "noncommutative quantum mechanics" displays an extra noncommutative structure in both the configurational and momentum sectors and has played an important role in some recent approaches to quantum cosmology [3, 4, 7, 10] and quantum gravity [14, 30]. Furthermore, the relation of the approach of [13] with the deformation formulation of noncommutative quantum mechanics [1, 2] was studied in [12].

In this paper we intend to further study the structure and the properties of the phase space formulation of quantum mechanics, focusing, this time, on the most central (and simplest) Schrödinger representation

$$x \longrightarrow \hat{X} = x \quad , \quad \xi_x \longrightarrow \hat{\Xi}_x = -i\frac{\partial}{\partial x}$$

(where the operators $\hat{X}, \hat{\Xi}_x$ act on phase space functions $\Psi(x, p) \in L^2(\mathbb{R}^{2n})$) and on its relation with the Moyal representation (1.3). In addition, we will

also discuss some features of the phase space formulation of mixed quantum states.

More precisely, we will:

(1) Present the phase space formulation of quantum mechanics in the Schrödinger representation.

(2) Determine and study the associated Weyl pseudo-differential calculus.

(3) Study the relation of the "Schrödinger phase space representation" with the "Schrödinger configuration space representation". In particular, show that there is a family of isometries $T_\chi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ (indexed by $\chi \in L^2(\mathbb{R}^n) : ||\chi|| = 1$) that intertwine an arbitrary configuration space operator $\hat{a}$ with the corresponding phase space operator $\hat{A}$.

(4) Show that the phase space formulation of quantum mechanics allows for an uniform treatment of both pure and mixed quantum states.

(5) Prove that the Schrödinger phase space representation and the Moyal phase space representation are unitarily related. Determine the one-parameter group of unitary transformations (and its infinitesimal generators) that connects the two representations and use it to prove the main spectral and dynamical results of the Moyal representation.

(6) Discuss the relation of the Moyal representation with the deformation quantization of Bayen *et al*.

1.1. Motivation: Double phase space formulation of classical mechanics.

Let us consider a dynamical system living on the phase space $\mathbb{R}^n \oplus \mathbb{R}^n$ spanned by the canonical variables (x, ξ_x) which satisfy the usual Poisson bracket structure $\{x, \xi_x\} = I$ (where I is the $n \times n$ identity matrix) all the others being zero. Let $h(x, \xi_x)$ be the Hamiltonian of the system.

A trivial formulation of this system in the double phase space is obtained by considering the extension $((p, \xi_p) \in \mathbb{R}^n \oplus \mathbb{R}^n)$

$$(x, \xi_x) \longrightarrow (x, p, \xi_x, \xi_p) : \left\{ \begin{array}{l} \{x, \xi_x\} = I \\ \{p, \xi_p\} = I \end{array} \right.$$

(all the other commutators being zero) and the new Hamiltonian

$$H_1(x, p, \xi_x, \xi_p) = h(x, \xi_x)$$

subjected to the initial data constraints

$$(1.7) \qquad\qquad p = p_0 \quad , \quad \xi_p = \xi_{p0}$$

The new set of observables

$$(1.8) \qquad\qquad A_1(x, p, \xi_x, \xi_p) = a(x, \xi_x)$$

yields exactly the same predictions as the original formulation in terms of the standard phase space observables $a(x, \xi_x)$. Notice that the role of the constraints here is only to fix the initial values of the non-physical sector of the theory. The constraints commute with the new Hamiltonian and are thus preserved through the time evolution.

Another double phase space formulation of classical mechanics is obtained from the action of the symplectic transformation

$$(1.9) \qquad S : \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \quad \begin{cases} x \longrightarrow x - \xi_p/2 \\ p \longrightarrow p - \xi_x/2 \\ \xi_x \longrightarrow \xi_x/2 + p \\ \xi_p \longrightarrow \xi_p/2 + x \end{cases}$$

on the previous double phase space formulation. The classical system is then described in terms of the new observables

$$A_2(x, p, \xi_x, \xi_p) = A_1(x - \xi_p/2, p - \xi_x/2, \xi_x/2 + p, \xi_p/2 + x)$$

and the new double phase space Hamiltonian

$$\begin{aligned} H_2(x, p, \xi_x, \xi_p) &= H_1(x - \xi_p/2, p - \xi_x/2, \xi_x/2 + p, \xi_p/2 + x) \\ &= h(x - \xi_p/2, \xi_x/2 + p) \end{aligned}$$

which yield, of course, the same physical predictions as the original double phase space formulation in terms of the Hamiltonian H_1 and the observables A_1.

Loosely speaking, this paper is devoted to presenting the quantum counterparts of the two former double phase space formulations of classical mechanics, to studying their properties and their relation with the standard, configuration space formulation of quantum mechanics. In spite of their apparently simple structure, they yield quantum theories which display a set of quite interesting properties, namely an uniform description of pure and mixed states and a remarkable connection with deformation quantization.

Before finishing this section, we note for future reference that the transformation S can be written as the composition of the two following symplectic maps

$$S = S_R(\pi/4) \circ S_D(\ln \sqrt{2})$$

where

$$S_D(s) : \mathbb{R}^{4n} \longrightarrow \mathbb{R}^{4n}$$

$$S_D(s)(x, p, \xi_x, \xi_p) = (e^s x, e^s p, e^{-s}\xi_x, e^{-s}\xi_p) \quad , \quad s \in \mathbb{R}$$

and $S_R(\theta)$ is a double rotation in the (x, ξ_p) and the (p, ξ_x) planes

$$S_R(\theta) : \mathbb{R}^{4n} \longrightarrow \mathbb{R}^{4n}$$

$$S_R(\theta) : \begin{cases} (x, \xi_p) \longrightarrow (x \cos\theta - \xi_p \sin\theta, x \sin\theta + \xi_p \cos\theta) \\ (p, \xi_x) \longrightarrow (p \cos\theta - \xi_x \sin\theta, p \sin\theta + \xi_x \cos\theta) \end{cases}$$

The infinitesimal generators of S_D and S_R are the Hamiltonians

$$(1.10) \qquad\qquad H_D = x \cdot \xi_x + p \cdot \xi_p$$

and

$$(1.11) \qquad\qquad H_R = -\xi_x \cdot \xi_p - x \cdot p$$

respectively.

1.2. **Notation.** A generic point of the original phase space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ is denoted by $z_x = (x, \xi_x)$ and that of the double phase space $\mathbb{R}^{4n} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ by $Z = (z_x, z_p)$ where $z_p = (p, \xi_p)$. We will also use $z = (x, p)$ and $z_0 = (x_0, \xi_{x0})$. The symplectic form on $\mathbb{R}^{2n}$ is $\sigma(z_x, z_x') = \xi_x \cdot x' - \xi_x' \cdot x$ and on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ is $\sigma_x \oplus \sigma_p(Z, Z') = \sigma(z_x, z_x') + \sigma(z_p, z_p')$. The corresponding symplectic groups are denoted by $Sp(2n, \sigma)$ and $Sp(4n, \sigma_x \oplus \sigma_p)$.

We write $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ for the dual space of tempered distributions. The notation $\langle a, t \rangle$ stands for the action of the distribution a on the test function t. Symbols (or classical observables) on $\mathbb{R}^{2n}$ are denoted by small Latin letters $a, b, ..$; if they have support on $\mathbb{R}^{4n}$ they are denoted by capital Latin letters $A, B, ...$. The wave functions in $L^2(\mathbb{R}^n)$ are denoted by small Greek letters $\psi, \phi, ...$ and those in $L^2(\mathbb{R}^{2n})$ by capital Greek letters $\Psi, \Phi, ...$. The standard inner product on $L^2(\mathbb{R}^n)$ is written $(\psi | \phi)$; on $L^2(\mathbb{R}^{2n})$ is $((\Psi | \Phi))$. The corresponding norms are $||\psi||$ and $|||\Psi|||$.

Operators acting on functions (or distributions) on $\mathbb{R}^n$ are usually denoted by small Latin letters with a hat $\hat{a}, \hat{b}, ...$ and those acting on phase space functions or distributions usually by $\hat{A}, \hat{B}, ...$ if they are in the Schrödinger representation and by $\tilde{A}, \tilde{B}, ...$ if they are in the Moyal representation. Weyl pseudo-differential operators display a W-superscript. The superscript $*$ denotes both the complex conjugation (for functions) and the adjoint (for operators).

The unitary Fourier transform and its inverse on $L^2(\mathbb{R}^n)$ are defined by

$$\hat{\phi}(p) = \mathcal{F}_x[\phi(x)](p) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot p} \phi(x) \, dx$$

and

$$\mathcal{F}_x^{-1}[\phi(p)](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot p} \phi(p) \, dp.$$

The same notation will also be used for the generalized (i.e. distributional) Fourier transform.

2. Phase space quantum mechanics in the Schrödinger representation: General results

The key object relating the configuration and the phase space Schrödinger representations of the Heisenberg algebra is the map

$$(2.1) \qquad T_\chi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n}); \quad \psi \to T_\chi[\psi] := \psi \otimes \chi^*$$

which is defined for some fixed $\chi \in L^2(\mathbb{R}^n)$ such that $||\chi|| = 1$. Since, for all $\phi, \psi \in L^2(\mathbb{R}^n)$

$$((T_\chi[\psi] | T_\chi[\phi])) = (\psi | \phi)$$

T_χ is an isometry from $L^2(\mathbb{R}^n)$ into a subspace $\mathcal{H}_\chi$ of $L^2(\mathbb{R}^{2n})$. Hence the map T_χ is also linear, injective and continuous. We then have the following

Definition 2.1 (States)**.** The states of the new (phase space) formulation of quantum mechanics are the wave functions

(2.2) $$\Psi(x,p) = T_\chi[\psi](x,p) = \psi(x)\chi^*(p)$$

which form the Hilbert space $\mathcal{H}_\chi = \mathrm{Ran}\, T_\chi \subset L^2(\mathbb{R}^{2n})$ (Corollary 2.5, bellow).

Notice that:

Remark 2.2. The choice of a particular χ plays here the same role as the imposition of the classical constraints (1.7) in the classical double phase space formulation. In fact, just like in the classical case, this imposition tantamount to the complete specification of the initial data for the unphysical sector of the theory. Notice also that the quantum states do not satisfy the strong (Dirac) version of the quantum constraints [23]

$$(\hat{p} - p_0)\Psi = 0 \quad , \quad (\hat{\xi}_p - \xi_{p0})\Psi = 0$$

which are, in fact, incompatible (because $\hat{p}, \hat{\xi}_p$ do not commute). They may, however satisfy a weaker version

$$((\Psi|(\hat{p} - p_0)\Psi)) = 0 \quad , \quad ((\Psi|(\hat{\xi}_p - \xi_{p0}))\Psi)) = 0$$

which may be seen as a necessary, but not sufficient, condition for the states to be of the form (2.2).

A key property of T_χ is given by:

Theorem 2.3. *The adjoint T_χ^* is the map*

$$T_\chi^* : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^n)$$

$$\Psi(x,p) \longrightarrow T_\chi^*[\Psi](x) = \int_{\mathbb{R}^n} \Psi(x,p)\chi(p)\,dp.$$

Proof. Consider the most general states

$$(\psi(x), \Psi(x,p)) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^{2n})$$

for which

(2.3) $$(\psi\,|\,\phi) = ((\Psi\,|\,T_\chi[\phi])), \quad \forall\phi \in L^2(\mathbb{R}^n).$$

Then:
(1) Ψ belongs to the domain of the adjoint $D(T_\chi^*)$ and
(2) $T_\chi^*[\Psi] = \psi$.

Eq.(2.3) is equivalent to

$$(\psi\,|\,\phi) = ((\Psi\,|\,\phi \otimes \chi^*)), \quad \forall\phi \in L^2(\mathbb{R}^n)$$

$$\Longleftrightarrow \int_{\mathbb{R}^n} \phi(x)\psi^*(x)\,dx = \int_{\mathbb{R}^n} \phi(x)\left[\int_{\mathbb{R}^n} \Psi^*(x,p)\chi^*(p)\,dp\right] dx, \quad \forall\phi \in L^2(\mathbb{R}^n)$$

$$\Longleftrightarrow \psi(x) = \int_{\mathbb{R}^n} \Psi(x,p)\chi(p)\,dp$$

It also follows that $\psi \in L^2(\mathbb{R}^n)$ if $\Psi \in L^2(\mathbb{R}^{2n})$ and so $D(T_\chi^*) = L^2(\mathbb{R}^{2n})$, which concludes the proof. $\qquad\square$

Remark 2.4. Let $\mathcal{H}_\chi =$ Ran T_χ. The restriction

$$T_\chi^* : \mathcal{H}_\chi \longrightarrow L^2(\mathbb{R}^n)$$

is a one to one map and is the inverse of T_χ. The proof is trivial.

Moreover:

Corollary 2.5. *The operator*

$$P_\chi : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n}), \quad P_\chi := T_\chi \circ T_\chi^*$$

is a projector and

$$\mathcal{H}_\chi = Ran\, T_\chi = Ran\, P_\chi$$

is a Hilbert space, Hilbert subspace of $L^2(\mathbb{R}^{2n})$, with the inner product

$$(2.4) \qquad\qquad ((T_\chi[\phi] \,|\, T_\chi[\psi])) = (\phi \,|\, \psi)$$

Proof. P_χ is an orthogonal projector because (i) $P_\chi = P_\chi^*$ and (ii) $P_\chi P_\chi = P_\chi$. The first identity follows from $T_\chi^{**} = T_\chi$:

$$((\Phi \,|\, P_\chi[\Psi])) = ((\Phi \,|\, T_\chi T_\chi^*[\Psi])) = (T_\chi^*[\Phi] \,|\, T_\chi^*[\Psi])$$

$$= ((T_\chi T_\chi^*[\Phi] \,|\, \Psi)) = ((P_\chi[\Phi] \,|\, \Psi)) \quad , \quad \forall \Phi, \Psi \in L^2(\mathbb{R}^{2n})$$

To prove (ii) just notice that $T_\chi^* T_\chi$ is the identity on $L^2(\mathbb{R}^n)$.

Since the range of T_χ^* is $L^2(\mathbb{R}^n)$ which is the domain of T_χ (and $P_\chi = T_\chi T_\chi^*$) we have Ran $T_\chi =$ Ran P_χ. On the other hand, the range of a projector in a Hilbert space is a closed linear subspace of that Hilbert space, thus also a Hilbert space. Hence, $\mathcal{H}_\chi =$ Ran P_χ is a Hilbert space, subspace of $L^2(\mathbb{R}^{2n})$ and with the same inner product. The identity (2.4) follows trivially . $\qquad\square$

Remark 2.6. Notice that there are many possible choices for the space of states $\mathcal{H}_\chi$. This is related, of course, to the possibility of choosing χ arbitrarily in $L^2(\mathbb{R}^n)$. However, these (different) phase space formulations are related by the one to one, orthogonal transformations

$$T_{\chi_1} T_{\chi_2}^* : \mathcal{H}_{\chi_2} \longrightarrow \mathcal{H}_{\chi_1}$$

and are thus completely equivalent.

We now introduce operators on $\mathcal{H}_\chi$:

Definition 2.7 (Phase space representation)**.** For each "configuration space" operator

$$\hat{a} : D(\hat{a}) \subset L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

we define an operator on $\mathcal{H}_\chi$

$$(2.5) \qquad\qquad \hat{A} : D(\hat{A}) \subset \mathcal{H}_\chi \longrightarrow \mathcal{H}_\chi$$

given by

$$D(\hat{A}) = T_\chi[D(\hat{a})]$$

$$\hat{A}\Psi = T_\chi \hat{a} T_\chi^* \Psi \quad , \quad \forall \Psi \in D(\hat{A})$$

and call $\hat{A}$ the "phase space operator associated with $\hat{a}$" or, more precisely, the "$\mathcal{H}_\chi$-representation of $\hat{a}$".

Remark 2.8. Notice that for

$$\hat{a} = \hat{x}_i = \text{multiplication by } x_i \text{ in } L^2(\mathbb{R}^n)$$

the associated operator is

$$\hat{A} = \hat{X}_i = T_\chi \hat{x}_i T_\chi^* = \text{multiplication by } x_i \text{ in } \mathcal{H}_\chi \subset L^2(\mathbb{R}^{2n})$$

and for

$$\hat{a} = \hat{\xi}_i = -i\partial_{x_i} \quad (\text{in } L^2(\mathbb{R}^n))$$

we have

$$\hat{A} = \hat{\Xi}_i = T_\chi \hat{\xi}_i T_\chi^* = -i\partial_{x_i} \quad (\text{in } \mathcal{H}_\chi \subset L^2(\mathbb{R}^{2n})).$$

Hence, each map T_χ generates a phase space Schrödinger representation of the Heisenberg algebra from the original configuration space Schrödinger representation.

We proceed with the study of some properties of the operators $\hat{A}$.

Theorem 2.9. *Let $\hat{A}$ be the phase space operator associated with $\hat{a}$ in the sense of the Definition 2.7. Then*
   (1) $\hat{A}$ is symmetric iff $\hat{a}$ is symmetric.
   (2) $\hat{A}$ is self-adjoint (as an operator in $\mathcal{H}_\chi$) iff $\hat{a}$ is self-adjoint.

Proof. 1) From

$$\hat{A}\Psi = T_\chi \hat{a} T_\chi^* T_\chi[\psi] = T_\chi[\hat{a}\psi], \quad \forall \Psi = T_\chi[\psi] \in D(\hat{A})$$

it follows that

$$((\hat{A}\Psi \,|\, \Phi)) = ((\Psi \,|\, \hat{A}\Phi)), \quad \forall \Psi, \Phi \in D(\hat{A})$$
$$\Longleftrightarrow ((T_\chi[\hat{a}\psi] \,|\, T_\chi[\phi])) = ((T_\chi[\psi] \,|\, T_\chi[\hat{a}\phi])), \quad \forall \psi, \phi \in D(\hat{a})$$
$$\Longleftrightarrow (\hat{a}\psi \,|\, \phi) = (\psi \,|\, \hat{a}\phi), \quad \forall \psi, \phi \in D(\hat{a})$$

and so $\hat{A}$ is symmetric in $D(\hat{A})$ iff $\hat{a}$ is symmetric in $D(\hat{a})$.

2) Let $\hat{a}$ be defined on a dense domain $D(\hat{a}) \subset L^2(\mathbb{R}^n)$ and consider the most general solution $(\psi, \xi) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ of

(2.6) $$(\xi \,|\, \phi) = (\psi \,|\, \hat{a}\phi), \quad \forall \phi \in D(\hat{a})$$

Then $\psi \in D(\hat{a}^*)$ and $\hat{a}^*\psi = \xi$. For $\Psi = T_\chi[\psi] \Longleftrightarrow \psi = T_\chi^*[\Psi]$ and $\Xi = T_\chi[\xi] \Longleftrightarrow \xi = T_\chi^*[\Xi]$, the previous equation is equivalent to

$$((\Xi \,|\, \Phi)) = ((\Psi \,|\, \hat{A}\Phi)), \quad \forall \Phi \in D(\hat{A})$$

and so (since $D(\hat{A}) = T_\chi[D(\hat{a})] \subset \mathcal{H}_\chi$ is also dense) $\Psi \in D(\hat{A}^*)$ and $\hat{A}^*\Psi = \Xi$. Hence

$$D(\hat{A}^*) = T_\chi[D(\hat{a}^*)] \quad , \quad \hat{A}^* = T_\chi \hat{a}^* T_\chi^*.$$

It follows that $\hat{a} = \hat{a}^* \Longleftrightarrow \hat{A} = \hat{A}^*$ as claimed. $\qquad \square$

Theorem 2.10 (Spectral results)**.** *The state $\Psi_\lambda \in \mathcal{H}_\chi$ is a solution of the eigenvalue equation*

(2.7) $$\hat{A}\Psi_\lambda = \lambda\Psi_\lambda$$

iff $\Psi_\lambda = T_\chi[\psi_\lambda]$ for some $\psi_\lambda \in L^2(\mathbb{R}^n)$ satisfying

(2.8) $$\hat{a}\psi_\lambda = \lambda\psi_\lambda$$

Hence, the two operators $\hat{A}$ and $\hat{a}$ display the same spectrum and their eigenfunctions are related by the T_χ-transformation.

Proof. The proof is trivial. In one direction it follows from applying T_χ to both sides of the eigenvalue equation (2.8) and, in the other direction, from applying T_χ^* to both sides of (2.7). We also need to notice that

(2.9) $$T_\chi[\hat{a}\psi] = \hat{A}T_\chi[\psi] = \hat{A}\Psi$$

and

(2.10) $$T_\chi^*[\hat{A}\Psi] = \hat{a}T_\chi^*[\Psi] = \hat{a}\psi.$$

$\square$

Corollary 2.11. *We conclude that the physical predictions of the phase space and the configuration space formulations of quantum mechanics are the same. For the average values we have from Definitions 2.1 and 2.7*

$$((\Psi \,|\, \hat{A}\Phi)) = (\psi \,|\, \hat{a}\phi)$$

and for the transition amplitudes (and probability amplitudes) from Theorem 2.10

$$((\Psi \,|\, \Psi_\lambda)) = (\psi \,|\, \psi_\lambda).$$

The same conclusion is valid for the dynamics,

Theorem 2.12 (Dynamics)**.** *Let $\hat{a} : D(\hat{a}) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be self-adjoint. Then $\psi(\cdot, t) \in L^2(\mathbb{R}^n)$ is the solution of the initial value problem*

$$i\frac{\partial\psi}{\partial t} = \hat{a}\psi \quad , \quad \psi(\cdot, 0) = \psi_0 \in D(\hat{a})$$

if and only if $\Psi = T_\chi[\psi]$ is the solution of

$$i\frac{\partial\Psi}{\partial t} = \hat{A}\Psi \quad , \quad \Psi(\cdot, 0) = T_\chi[\psi_0] \in D(\hat{A}).$$

Proof. This theorem is also trivial. Again the equivalence of the two dynamics can be proved by applying the maps T_χ and T_χ^* to the dynamical equations and by taking into account the relations (2.9, 2.10) and also that the time derivative commutes with T_χ (because χ is time independent)

$$\frac{\partial}{\partial t}T_\chi[\psi] = T_\chi[\frac{\partial}{\partial t}\psi]$$

and with T_χ^*

$$\frac{\partial}{\partial t}T_\chi^*[\Psi] = T_\chi^*[\frac{\partial}{\partial t}\Psi]$$

$\square$

3. Mixed states in phase space quantum mechanics

The formalism of the last section was defined in the spaces $\mathcal{H}_\chi$ and provides a description of pure states, only. However, one realizes that most of the results can be generalized for the case where the space of states is a Hilbert space larger than a particular $\mathcal{H}_\chi$ or is even the entire $L^2(\mathbb{R}^{2n})$.

Such generalizations lead to a suitable description of mixed states as we now show. More precisely, we will discuss (some aspects of) the extension of the formalism of the last section to the case where the Hilbert space of states is of the form

$$\mathcal{H} = \oplus_k \mathcal{H}_{\chi_k}$$

for some orthogonal, finite set of functions $\chi_k \in L^2(\mathbb{R}^n)$ such that

$$\sum_k ||\chi_k||^2 = 1.$$

Then

$$\Psi \in \mathcal{H} \implies \Psi = \sum_k \psi_k \otimes \chi_k^*, \quad \psi_k \in L^2(\mathbb{R}^n)$$

The normalization of Ψ follows directly from the normalization of each ψ_k

$$|||\Psi|||^2 = ((\Psi \,|\, \Psi)) = \sum_k (\psi_k \,|\, \psi_k)(\chi_k \,|\, \chi_k) = \sum_k ||\psi_k||^2 ||\chi_k||^2$$

and this formula suggests that the square of the norm of χ_k can be interpreted as the classical probability associated with the k-component of the mixed state Ψ.

The $\mathcal{H}$-representation of a configuration space operator $\hat{a}$ generalizes Definition 2.7. We now have

$$\hat{A} : D(\hat{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

where the domain is

$$D(\hat{A}) = \oplus_k T_{\chi_k}[D(\hat{a})]$$

and

$$\hat{A}\Psi = \sum_k T_{\chi_k} \hat{a} T_{\chi_k}^* \Psi \quad , \quad \forall \Psi \in D(\hat{A}).$$

Let us now recover the quantum predictions for mixed states from applying the standard rules of "pure state quantum mechanics" to the phase space $\mathcal{H}$-formulation. Let ϕ_α be a normalized eigenfunction of $\hat{a}$ associated with the non-degenerate eigenvalue α (to make it simpler we shall assume the one-dimensional case; $n = 1$). Then α is also an eigenvalue of $\hat{A}$ (in this case, degenerate). A normalized basis of the α-eigenspace of $\hat{A}$ is given by the eigenfunctions

$$\Phi_{\alpha,k} = \phi_\alpha \otimes \frac{\chi_k^*}{||\chi_k||},$$

hence the probability associated with the eigenvalue α is (from standard rules)

$$\mathcal{P}(\hat{A} = \alpha) = \sum_k |((\Psi \,|\, \Phi_{\alpha,k}))|^2 = \sum_k |(\psi_k \,|\, \phi_\alpha)|^2 \,||\chi_k||^2$$

which is then a convex combination of the probabilities $\mathcal{P}(\hat{a} = \alpha)$ calculated for each of the components ψ_k of the mixed state Ψ. This result thus re-enforces the interpretation of the square of the norm of χ_k as the classical probability associated with the k-component of the mixed state.

The projection of Ψ into the α-eigenspace of $\hat{A}$ yields the state

$$\Upsilon = \sum_k ((\Psi \,|\, \Phi_{\alpha,k}))\Phi_{\alpha,k} = \sum_k (\psi_k \,|\, \phi_\alpha)\phi_\alpha \otimes \chi_k^*$$

and so the collapse of the wave function (produced by a measurement of $\hat{A}$ with output α) yields

$$\Upsilon_c = \frac{\Upsilon}{|||\Upsilon|||}.$$

Just like for standard, pure state, quantum mechanics the probability $\mathcal{P}(\hat{A} = \alpha)$ is identical to the transition probability $|((\Psi \,|\, \Upsilon_c))|^2$. Indeed

$$|((\Psi \,|\, \Upsilon_c))|^2 = \frac{1}{|||\Upsilon|||^2}|((\sum_k \psi_k \otimes \chi_k^* \,|\, \sum_j (\psi_j \,|\, \phi_\alpha)\phi_\alpha \otimes \chi_j^*))|^2$$

and since

$$|||\Upsilon|||^2 = \sum_k |(\psi_k \,|\, \phi_a)|^2 ||\chi_k||^2$$

we get

$$|((\Psi \,|\, \Upsilon_c))|^2 = \frac{1}{|||\Upsilon|||^2} \left| \sum_k |(\psi_k \,|\, \phi_a)|^2 ||\chi_k||^2 \right|^2 = \sum_k |(\psi_k \,|\, \phi_a)|^2 ||\chi_k||^2.$$

Hence, a representation of mixed states in terms of standard wave functions with support on the phase space is possible. We intend to study this topic in more detail in a forthcoming paper.

4. Phase space Weyl calculus

We start this section with a brief review of the standard definitions and properties of Weyl operators acting on functions with support on the configuration space (for details and proofs the interested reader may refer to [17, 19, 24, 27, 28, 32]). We then present the extension of these operators to phase space functions and study the main properties of the resulting phase space Weyl calculus.

4.1. **Standard Weyl calculus.** Let $\mathscr{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ be the space of linear and continuous operators of the form $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$. In view of Schwartz kernel theorem all operators $\hat{a} \in \mathscr{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ admit a kernel representation

$$(4.1) \qquad \hat{a}\psi(x) = \langle K_a(x, \cdot), \psi(\cdot) \rangle$$

where $K_a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. The Weyl symbol of $\hat{a}$ is then

$$(4.2) \qquad a(x, \xi_x) = \int_{\mathbb{R}^n} e^{-i\xi_x \cdot y} K_a(x + \frac{1}{2}y, x - \frac{1}{2}y) \, dy$$

and, conversely,

$$(4.3) \qquad K_a(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\xi \cdot (x-y)} a\left(\frac{x+y}{2}, \xi\right) d\xi$$

where the integrals are interpreted as generalized Fourier (and inverse Fourier) transforms (i.e. in the sense of distributions). The inverse formula can be re-written in the form

$$(4.4) \qquad K_a(x, y) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{\frac{i}{2}\xi \cdot (x+y)} \mathcal{F}_\sigma a(x - y, \xi) \, d\xi$$

where $\mathcal{F}_\sigma$ is the symplectic Fourier transform (1.2).

An important property of Weyl operators is that, if $\hat{b} \in \mathscr{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$ then $\hat{c} = \hat{a}\hat{b} \in \mathscr{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ and its Weyl symbol is given by

$$(4.5)$$
$$c(z_x) = \left(\frac{1}{4\pi}\right)^{2n} \int \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{\frac{i}{2}\sigma(u,v)} a(z_x + \frac{1}{2}u) b(z_x - \frac{1}{2}v) \, du \, dv = a(z_x) \star b(z_x)$$

where $\star$ is the "twisted product" or Moyal product familiar from deformation quantization [5, 25, 32].

The Weyl correspondence $a \overset{\text{Weyl}}{\longleftrightarrow} \hat{a}^W$ (given by eqs.(4.1,4.2)) can be written more straightforwardly as

$$(4.6) \qquad \hat{a}^W = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0) \hat{T}(z_0) \, dz_0$$

where the integral is an operator valued (Bochner) integral and $\hat{T}(z_0) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is the Heisenberg-Weyl operator ($z_0 = (x_0, \xi_{x0})$)

$$(4.7) \qquad \hat{T}(z_0) = e^{-i\sigma(\hat{z}_x, z_0)} = e^{-i(x_0 \cdot \hat{\xi}_x - \xi_{x0} \cdot \hat{x})};$$

explicitly:

$$(4.8) \qquad \hat{T}(x_0, \xi_{x0})\psi(x) = e^{i(\xi_{x0} \cdot x - \frac{1}{2}\xi_{x0} \cdot x_0)} \psi(x - x_0).$$

Remark 4.1. The Heisenberg-Weyl operators $\hat{T}(z_0)$ are unitary operators associated with the self-adjoint Hamiltonians $\sigma(\hat{z}_x, z_0)$. In fact, one can easily check that

$$\psi(x, t) = \hat{U}(z_0, t)\psi_0(x) = e^{it(\xi_{x0} \cdot x - \frac{1}{2}\xi_{x0} \cdot x_0)} \psi_0(x - tx_0), \quad \psi_0 \in \mathcal{S}(\mathbb{R}^n)$$

is the unique solution of the initial value problem

$$i\frac{\partial}{\partial t}\psi = \sigma(\hat{z}_x, z_0)\psi \quad , \quad \psi(\cdot, 0) = \psi_0 \in \mathcal{S}(\mathbb{R}^n)$$

Hence $\hat{U}(z_0, t) = e^{-it\sigma(\hat{z}_x, z_0)}$ in $\mathcal{S}(\mathbb{R}^n)$ and since both operators are continuous we also have

$$\hat{U}(z_0, t) = e^{-it\sigma(\hat{z}_x, z_0)} \quad \text{in } L^2(\mathbb{R}^n).$$

Finally, it is trivial to check that $\hat{T}(z_0) = \hat{U}(z_0, 1)$.

The Weyl correspondence satisfies the metaplectic covariance property, which will be useful in section 5,

Theorem 4.2. *Let* $\mathrm{Mp}(2n, \sigma)$ *be the metaplectic group, i.e. the unitary representation of the double cover of* $Sp(2n, \sigma)$. *Let* $\hat{a}^W \overset{Weyl}{\longleftrightarrow} a$, *let* $S \in Sp(2n, \sigma)$ *and let* $\hat{S} \in \mathrm{Mp}(2n, \sigma)$ *be (one of the two) metaplectic operators that projects onto* S. *Then*

(4.9)
$$\hat{S}^{-1}\hat{a}^W \hat{S} \overset{Weyl}{\longleftrightarrow} a \circ S$$

For proof see [28, 32].

4.2. **Weyl calculus in phase space.** In this section we consider the extensions of $\hat{a}^W$ to $\mathcal{H}_\chi$ and $\mathcal{S}(\mathbb{R}^{2n})$. We will focus on the case where $\chi \in \mathcal{S}(\mathbb{R}^n)$ so that $T_\chi[\psi] \in \mathcal{S}(\mathbb{R}^{2n})$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$.

Let us then introduce the notation

$$\mathcal{S}_\chi = \mathcal{S}(\mathbb{R}^{2n}) \cap \mathcal{H}_\chi = T_\chi[\mathcal{S}(\mathbb{R}^n)]$$

and consider the extension of T_χ to $\mathcal{S}'(\mathbb{R}^n)$

$$T_\chi : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n}); \psi \longrightarrow T_\chi[\psi] = \psi \otimes \chi^*$$

Then $\Psi = T_\chi[\psi]$ is the distribution

$$\langle \Psi, t \rangle = \langle \psi, T_{\chi^*}^*[t] \rangle, \quad \forall t \in \mathcal{S}(\mathbb{R}^{2n})$$

which is well defined because $\chi \in \mathcal{S}(\mathbb{R}^n)$.

By a natural generalization of Definition 2.7 (to operators of the form $\hat{a} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$) the extension of $\hat{a}^W$ to $\mathcal{S}_\chi \subset \mathcal{S}(\mathbb{R}^{2n})$ is given by

$$\hat{A}_\chi^W : \mathcal{S}_\chi \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

$$\hat{A}_\chi^W = T_\chi \hat{a}^W T_\chi^* = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0) T_\chi \hat{T}(z_0) T_\chi^* \, dz_0$$

and its action is explicitly

$$\hat{A}_\chi^W \Psi(x, p) = \left(\frac{1}{2\pi}\right)^n \langle \mathcal{F}_\sigma a(\cdot), T_\chi \hat{T}(\cdot) T_\chi^* \Psi(x, p) \rangle$$

which is well defined since

$$T_\chi \hat{T}(x_0, \xi_{x0}) T_\chi^* \Psi(x, p) = e^{i(\xi_{x0} \cdot x - \frac{1}{2}\xi_{x0} \cdot x_0)} \Psi(x - x_0, p)$$

belongs to $\mathcal{S}(\mathbb{R}^{2n})$ for all (x, p).

Now consider the following

Definition 4.3. Let the *phase space Heisenberg-Weyl operator* be defined by

$$\hat{T}_{PS}(z_0) : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n})$$

(4.10) $$\hat{T}_{PS}(z_0)\Psi(x,p) := e^{i(\xi_{x0} \cdot x - \frac{1}{2}\xi_{x0} \cdot x_0)}\Psi(x - x_0, p).$$

Notice that

Remark 4.4. From Remark 4.1 it is trivial to conclude that

$$\hat{T}_{PS}(z_0) = \hat{U}_{PS}(z_0, t)\Big|_{t=1}$$

where $(\hat{Z}_x = (\hat{X}, \hat{\Xi}_x)$, cf. Remark 2.8)

$$\hat{U}_{PS}(z_0, t) = e^{-it\sigma(\hat{Z}_x, z_0)}$$

is the one-parameter unitary evolution group with infinitesimal generator $\sigma(\hat{Z}_x, z_0)$. Moreover, from the definition of $\hat{T}_{PS}(z_0)$ we also realize that

$$\hat{T}_{PS}(z_0)\Big|_{\mathcal{S}_\chi} = T_\chi \hat{T}(z_0) T_\chi^*\Big|_{\mathcal{S}_\chi}.$$

It follows that $\hat{A}_\chi^W$ can be written as

(4.11) $$\hat{A}_\chi^W = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0)\hat{T}_{PS}(z_0)\, dz_0.$$

Now note that the functional form of the previous operator is independent of χ and that its action can be consistently extended to $\mathcal{S}(\mathbb{R}^{2n})$. This naturally suggests

Definition 4.5. Let the *phase space Weyl operator* $\hat{A}^W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$, associated with the Weyl symbol $a(x, \xi_x)$, be defined by

(4.12) $$\hat{A}^W = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0)\hat{T}_{PS}(z_0)\, dz_0$$

and let us denote by $a \overset{\text{Weyl}}{\longleftrightarrow} \hat{A}^W$ (or, more precisely, $A \overset{\text{Weyl}}{\longleftrightarrow} \hat{A}^W$; see the next Theorem) the *phase space Weyl correspondence* between a (or A) and $\hat{A}^W$.

We now prove that $\hat{A}^W$ is indeed a Weyl operator, whose restriction to $\mathcal{S}_\chi$ satisfies $\hat{A}^W|_{\mathcal{S}_\chi} = \hat{A}_\chi^W$ for all $\chi \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 4.6. Let $a(x, \xi_x) \in \mathcal{S}'(\mathbb{R}^{2n})$ be the Weyl symbol of $\hat{a}^W$. Then the operator $\hat{A}^W : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ defined by

(4.13) $$\hat{A}^W\Psi = \left(\frac{1}{2\pi}\right)^n \langle \mathcal{F}_\sigma a(\cdot), \hat{T}_{PS}(\cdot)\Psi \rangle$$

that is, formally, by eq.(4.12), is a Weyl operator with symbol $A = a \otimes 1$, in coordinates

(4.14) $$A(x, p, \xi_x, \xi_p) = a(x, \xi_x)$$

and so $A \in \mathcal{S}'(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n})$.

Proof. Let $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$. Since $\hat{T}_{PS}\Psi(z) \in \mathcal{S}(\mathbb{R}^{2n})$ for all z and $\mathcal{F}_\sigma a \in \mathcal{S}'(\mathbb{R}^{2n})$ the operator $\hat{A}^W$, given by eq.(4.13), is well-defined. We then have

$$
\begin{aligned}
\hat{A}^W \Psi(x,p) &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{F}_\sigma a(x_0, \xi_{x0}) e^{i(\xi_{x0}\cdot x - \frac{1}{2}\xi_{x0}\cdot x_0)} \Psi(x - x_0, p)\, dx_0 d\xi_{x0} \\
&= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathcal{F}_\sigma a(x - x', \xi_{x0}) e^{\frac{i}{2}(\xi_{x0}\cdot x + \xi_{x0}\cdot x')} \Psi(x', p)\, dx' d\xi_{x0}
\end{aligned}
$$

where we performed the substitution $x' = x - x_0$. Hence, the action of $\hat{A}^W$ can be written

$$
\hat{A}^W \Psi(x,p) = \langle K_A(x,p;x',p'), \Psi(x',p')\rangle
$$

for the kernel $K_A \in \mathcal{S}'(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ given by

$$
K_A(x,p;x',p') = \left(\frac{1}{2\pi}\right)^n \delta(p - p') \int_{\mathbb{R}^n} \mathcal{F}_\sigma a(x - x', \xi_{x0}) e^{\frac{i}{2}(\xi_{x0}\cdot x + \xi_{x0}\cdot x')}\, d\xi_{x0}
$$

where the integral is interpreted in the distributional sense. Comparing this expression with eq.(4.4) we find that

$$
K_A(x,p;x',p') = K_a(x;x')\delta(p - p')
$$

where $K_a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is the kernel of the Weyl operator $\hat{a}^W$.

From eq.(4.2) it follows that the Weyl symbol of $\hat{A}^W$ is

$$
\begin{aligned}
A(x,p,\xi_x,\xi_p) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} K_A(x + \tfrac{1}{2}\eta_x, p + \tfrac{1}{2}\eta_p; x - \tfrac{1}{2}\eta_x, p - \tfrac{1}{2}\eta_p) e^{-i(\xi_x \cdot \eta_x + \xi_p \cdot \eta_p)}\, d\eta_x d\eta_p \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} K_a(x + \tfrac{1}{2}\eta_x; x - \tfrac{1}{2}\eta_x)\delta(\eta_p) e^{-i(\xi_x \cdot \eta_x + \xi_p \cdot \eta_p)}\, d\eta_x d\eta_p \\
&= \int_{\mathbb{R}^n} K_a(x + \tfrac{1}{2}\eta_x; x - \tfrac{1}{2}\eta_x) e^{-i\xi_x \cdot \eta_x}\, d\eta_x = a(x, \xi_x)
\end{aligned}
$$

$\square$

Theorem 4.7. *The operator $\hat{A}^W$ satisfies*

(4.15)
$$
\hat{A}^W\Big|_{\mathcal{S}_\chi} = \hat{A}^W_\chi
$$

and we have the intertwining relations

(4.16)
$$
\hat{A}^W T_\chi = T_\chi \hat{a}^W
$$

and

(4.17)
$$
T_\chi^* \hat{A}^W = \hat{a}^W T_\chi^*
$$

Proof. The identity $\hat{A}^W\Big|_{\mathcal{S}_\chi} = \hat{A}^W_\chi$ is a direct consequence of eqs.(4.11,4.12) and $\mathcal{S}_\chi \subset \mathcal{S}(\mathbb{R}^{2n})$.

Let us prove the first intertwining relation. For every $\psi \in D(\hat{a}^W) = \mathcal{S}(\mathbb{R}^n)$ we have $T_\chi[\psi] \in \mathcal{S}_\chi$ and so

$$\hat{A}^W T_\chi \psi = \hat{A}^W_\chi T_\chi \psi = T_\chi \hat{a}^W T_\chi^* T_\chi \psi = T_\chi \hat{a}^W \psi$$

where we used the identity $T_\chi^* T_\chi = 1$ in $\mathcal{S}(\mathbb{R}^n)$.

The second intertwining relation follows from

(4.18) $$T_\chi^* \hat{T}_{PS}(z_0) = \hat{T}(z_0) T_\chi^*$$

by interchanging the (Bochner) integrals (eqs.(4.6) and (4.12)) with the operator T_χ^*. The proof of eq.(4.18) in $\mathcal{S}(\mathbb{R}^{2n})$ is straightforward

$$
\begin{aligned}
T_\chi^* \hat{T}_{PS}(z_0)\Psi(x,p) &= T_\chi^* e^{i(\xi_{x0}\cdot x - \frac{1}{2}\xi_{x0}\cdot x_0)}\Psi(x-x_0,p) \\
&= \int e^{i(\xi_{x0}\cdot x - \frac{1}{2}\xi_{x0}\cdot x_0)}\Psi(x-x_0,p)\chi(p)\,dp \\
&= \hat{T}(z_0)\int \Psi(x,p)\chi(p)\,dp = \hat{T}(z_0)T_\chi^*\Psi(x,p).
\end{aligned}
$$

$\square$

The spectral results for the operators $\hat{A}^W$ follow from the ones for $\hat{a}^W$ by a direct application of the previous Theorem.

Corollary 4.8. *Let $\hat{a}^W$ and $\hat{A}^W$ be the Weyl and phase space Weyl operators associated with the symbol $a \in \mathcal{S}(\mathbb{R}^{2n})$, respectively. Then*

(i) The eigenvalues of $\hat{a}^W$ and $\hat{A}^W$ are the same.

(ii) Let $\chi \in \mathcal{S}(\mathbb{R}^n)$. If ψ_λ is an eigenfunction of $\hat{a}^W$ then $\Psi_\lambda = T_\chi[\psi_\lambda]$ is an eigenfunction of $\hat{A}^W$ (associated with the same eigenvalue).

(iii) Conversely, let Ψ_λ be an eigenfunction of $\hat{A}^W$. If $\psi_\lambda = T_\chi^*[\Psi_\lambda] \neq 0$ then ψ_λ is an eigenfunction of $\hat{a}^W$ (associated with the same eigenvalue).

Proof. Let $\chi \in \mathcal{S}(\mathbb{R}^n)$ and let ψ_λ be an eigenfunction of $\hat{a}^W$

$$\hat{a}^W \psi_\lambda = \lambda \psi_\lambda.$$

Then $\Psi_\lambda = T_\chi[\psi_\lambda]$ is an eigenfunction of $\hat{A}^W_\chi$ (associated with the same eigenvalue)

$$\hat{A}^W \Psi_\lambda = \hat{A}^W T_\chi[\psi_\lambda] = T_\chi \hat{a}^W \psi_\lambda = \lambda T_\chi[\psi_\lambda] = \lambda \Psi_\lambda$$

where we used eq.(4.16). This proves (ii).

Conversely, assume that

$$\hat{A}^W \Psi_\lambda = \lambda \Psi_\lambda$$

and let $\chi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\psi_\lambda = T_\chi^*[\Psi_\lambda] \neq 0$. Then

$$\hat{a}^W \psi_\lambda = \hat{a}^W T_\chi^* \Psi_\lambda = T_\chi^* \hat{A}^W \Psi_\lambda = \lambda T_\chi^* \Psi_\lambda = \lambda \psi_\lambda$$

where we used eq.(4.17). Hence ψ_λ is also an eigenfunction of $\hat{a}^W$ (associated with the same eigenvalue) which proves (iii).

To conclude the proof of (i) we just notice that for every $\Psi_\lambda \in \mathcal{S}(\mathbb{R}^{2n}) - \{0\}$ there is always some $\chi \in \mathcal{S}(\mathbb{R}^n)$ such that $T_\chi^*[\Psi_\lambda] \neq 0$. $\square$

5. Moyal representation

In this section we construct another phase space representation of quantum mechanics, which is intimately connected with the deformation quantization of Bayen et al [5, 6]. Namely, the eigenvalue equation (for a generic Weyl operator in this representation) is just the Moyal $\star$-genvalue equation (for the Weyl symbol of that operator) and its solutions are thus the $\star$-genfunctions (Theorem 5.8 and Corollary 5.9). Moreover, the Schrödinger equation (in this representation) can be written in terms of the Moyal starproduct and the Weyl symbol of the Hamiltonian operator (Corollary 5.10). For these reasons we shall call it the *Moyal representation*. This formulation of quantum mechanics has been studied before (also for the more general case where the canonical structure is given by the extended Heisenberg algebra [13]) using a set of partial isometries $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$ (the windowed wavepacket transform, familiar from time-frequency analysis [22]) mapping the standard Schrödinger configuration space representation into the Moyal phase space representation [18, 19, 20, 21]. Here we will follow a different approach by showing that the Schrödinger phase space representation and the Moyal representation are related by the unitary (in fact metaplectic) transformation U associated with the symplectic transformation eq.(1.9). This result allows us to translate the Weyl pseudo-differential calculus and the spectral and dynamical results of the Schrödinger phase space representation directly into the Moyal representation.

In the next subsection we will determine the unitary transformation U explicitly and show that its action on $\Psi \in \mathcal{S}_{\hat{\chi}}$ is nothing else but the cross Wigner function associated with the density matrix element $|\psi\rangle\langle\chi|$. In subsection 5.2 we use the transformation U to determine the Weyl pseudo-differential calculus in the Moyal representation. Finally, in subsection 5.3 we construct the eigenvalue and dynamical equations in this representation and study their relation with the deformation quantization formulation.

5.1. **Unitary transformation.** In this section we determine the one-parameter quantum evolution groups generated by the operators

$$\hat{H}_D = \frac{1}{2}(\hat{X} \cdot \hat{\Xi}_x + \hat{\Xi}_x \cdot \hat{X}) + \frac{1}{2}(\hat{P} \cdot \hat{\Xi}_p + \hat{\Xi}_p \cdot \hat{P})$$

$$(5.1) \qquad \hat{H}_R = -\hat{\Xi}_x \cdot \hat{\Xi}_p - \hat{X} \cdot \hat{P}$$

which are obtained after quantizing the Hamiltonians (1.10,1.11) and use then to determine the explicit form of the unitary operator U. Note that

the operators $\hat{X}, \hat{P}, \hat{\Xi}_x, \hat{\Xi}_p$ are given in the phase space Schrödinger representation

(5.2) $$\hat{X} = x = (x_1, ..., x_n) \quad , \quad \hat{P} = p = (p_1, ..., p_n)$$

$$\hat{\Xi}_x = -i\partial_x = (-i\partial_{x1}, ..., -i\partial_{xn}), \quad \hat{\Xi}_p = -i\partial_p = (-i\partial_{p1}, ..., -i\partial_{pn})$$

Remark 5.1. Both $\hat{H}_D$ and $\hat{H}_R$ are self-adjoint on their maximal domains

$$D_{max}(\hat{H}_D) = \{\Psi \in L^2(\mathbb{R}^{2n}) : \hat{H}_D\Psi \in L^2(\mathbb{R}^{2n})\}$$

$$D_{max}(\hat{H}_R) = \{\Psi \in L^2(\mathbb{R}^{2n}) : H_R\Psi \in L^2(\mathbb{R}^{2n})\}.$$

This immediately follows from the fact that they are polynomial differential expressions which are formally self-adjoint [15]

We then have

Theorem 5.2. *The one-parameter quantum evolution group generated by $\hat{H}_D$ is given by $(s \in \mathbb{R})$*

(5.3) $$U_D(s) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n}); \quad U_D(s)\Psi(x,p) = e^{-ns}\Psi(e^{-s}x, e^{-s}p)$$

Proof. Since $\hat{H}_D$ is self adjoint and $\mathcal{S}(\mathbb{R}^{2n}) \subset D(\hat{H}_D)$

$$\Psi(x, p, s) = e^{-is\hat{H}_D}\Psi_0(x, p)$$

is the unique solution of (cf. [[26], chapter 5])

$$i\frac{\partial\Psi}{\partial s} = \hat{H}_D\Psi \quad , \quad \Psi(\cdot, 0) = \Psi_0 \in \mathcal{S}(\mathbb{R}^{2n}).$$

This equation reads

$$i\frac{\partial\Psi}{\partial s} = -i(x \cdot \partial_x + p \cdot \partial_p + n)\Psi$$

and it is trivial to check that

$$\Psi(x, p, s) = e^{-ns}\Psi_0(e^{-s}x, e^{-s}p)$$

is its explicit solution. Defining

$$U_D(s) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n}); \quad U_D(s)\Psi(x, p) := e^{-ns}\Psi(e^{-s}x, e^{-s}p)$$

one can also easily check that $U_D(s)$ is linear and unitary (so bounded and continuous) operator. Since

$$U_D(s)|_{\mathcal{S}(\mathbb{R}^{2n})} = e^{-is\hat{H}_D}\Big|_{\mathcal{S}(\mathbb{R}^{2n})}$$

and both operators are continuous (and $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $L^2(\mathbb{R}^{2n})$),

$$U_D(s) = e^{-is\hat{H}_D}.$$

$\square$

Theorem 5.3. *The one-parameter unitary evolution group generated by $\hat{H}_R$ is*

$$U_R(\theta) : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n})$$

(5.4) $$U_R(\theta)\Psi(x, p) := \mathcal{F}_p^{-1}\left[\hat{\Psi}(x\cos\theta + \xi_p\sin\theta, \xi_p\cos\theta - x\sin\theta)\right]$$

where

$$\hat{\Psi}(x, \xi_p) = \mathcal{F}_p[\Psi(x, p)](x, \xi_p) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi_p \cdot p}\Psi(x, p)\, dp$$

and

$$\mathcal{F}_p^{-1}[\Psi(x, \xi_p)](x, p) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i\xi_p \cdot p}\Psi(x, \xi_p)\, d\xi_p$$

are the partial and inverse partial Fourier transforms, defined as unitary operators on $L^2(\mathbb{R}^{2n})$.

Proof. Since $\hat{H}_R$ is self-adjoint and $\mathcal{S}(\mathbb{R}^{2n}) \subset D(\hat{H}_R)$, the unique solution of the initial value problem

(5.5) $$i\frac{\partial\Psi}{\partial\theta} = \hat{H}_R\Psi \quad, \quad \Psi(\cdot, 0) = \Psi_0 \in \mathcal{S}(\mathbb{R}^{2n})$$

is given by

(5.6) $$\Psi(x, p, \theta) = e^{-i\theta\hat{H}_R}\Psi_0(x, p)$$

Since

$$\hat{H}_R = \partial_x \cdot \partial_p - x \cdot p = \mathcal{F}_p^{-1}\left[i\xi_p \cdot \partial_x - ix \cdot \partial_{\xi_p}\right]\mathcal{F}_p$$

we conclude, defining $\hat{\Psi}(x, \xi_p, \theta) = \mathcal{F}_p[\Psi(x, p, \theta)]$, that $\hat{\Psi}$ satisfies the initial value problem

$$i\frac{\partial\hat{\Psi}}{\partial\theta} = \left[i\xi_p \cdot \partial_x - ix \cdot \partial_{\xi_p}\right]\hat{\Psi}, \quad \hat{\Psi}(\cdot, 0) = \hat{\Psi}_0 \in \mathcal{S}(\mathbb{R}^{2n})$$

whose solution

$$\hat{\Psi}(x, \xi_p, \theta) = \hat{\Psi}_0(x(\theta), \xi_p(\theta))$$
$$x(\theta) = x\cos\theta + \xi_p\sin\theta \quad, \quad \xi_p(\theta) = -x\sin\theta + \xi_p\cos\theta$$

always is in $\mathcal{S}(\mathbb{R}^{2n})$.

Hence, the unique solution of (5.5) is

$$\Psi(x, p, \theta) = \mathcal{F}_p^{-1}\left[\hat{\Psi}(x, \xi_p, \theta)\right] = \mathcal{F}_p^{-1}\left[\hat{\Psi}_0(x\cos\theta + \xi_p\sin\theta, -x\sin\theta + \xi_p\cos\theta)\right]$$

Defining

$$U_R(\theta) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$$
$$U_R(\theta)\Psi(x, p) := \mathcal{F}_p^{-1}\left[\hat{\Psi}(x\cos\theta + \xi_p\sin\theta, -x\sin\theta + \xi_p\cos\theta)\right]$$

it is trivial to check that $U_R(\theta)$ is linear, unitary (so bounded and continuous) operator. Since by (5.6)

$$U_R(\theta)|_{\mathcal{S}(\mathbb{R}^{2n})} = e^{-i\theta\hat{H}_R}\Big|_{\mathcal{S}(\mathbb{R}^{2n})}$$

and both operators are continuous (and $\mathcal{S}(\mathbb{R}^{2n})$ is dense in $L^2(\mathbb{R}^{2n})$),

$$U_R(\theta) = e^{-i\theta \hat{H}_R}.$$

$\square$

We are now in position to define the unitary operator that corresponds to the symplectic transformation eq.(1.9)

Corollary 5.4. *Consider the unitary operator*

$$U : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n})$$

defined by

(5.7) $$U := U_D^{-1}(\ln \sqrt{2})U_R^{-1}(\pi/4)$$

Then U acts as

(5.8) $$U\Psi(x,p) = \mathcal{F}_p^{-1}\left[\hat{\Psi}(x - \xi_p/2, x + \xi_p/2)\right](x,p)$$

and for

$$\Psi(x,p) = T_{\hat{\chi}}[\psi](x,p) = \psi(x)\hat{\chi}^*(p)$$

where $\hat{\chi}(p) = \mathcal{F}_{\xi_p}[\chi(\xi_p)](p)$ and $\hat{\chi}^*(p) = \overline{\hat{\chi}(p)}$, we have

(5.9) $$U\Psi(x,p) = \frac{1}{(2\pi)^{n/2}} \int e^{ip\cdot\xi_p}\psi(x - \frac{\xi_p}{2})\chi^*(x + \frac{\xi_p}{2})\,d\xi_p$$

which is (up to a factor of $(2\pi)^{n/2}$) the cross Wigner function associated with the density matrix element $|\psi\rangle\langle\chi|$, i.e. $UT_{\hat{\chi}}[\psi] = W(\psi, \chi)$.

Proof. We have

$$U_R^{-1}(\pi/4)\Psi(x,p) = U_R(-\pi/4)\Psi(x,p) = \mathcal{F}_p^{-1}\left[\hat{\Psi}(\frac{\sqrt{2}}{2}(x - \xi_p), \frac{\sqrt{2}}{2}(x + \xi_p))\right]$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi_p\cdot p}\hat{\Psi}(\frac{\sqrt{2}}{2}(x - \xi_p), \frac{\sqrt{2}}{2}(x + \xi_p))\,d\xi_p$$

and since for arbitrary $\Phi(x,p) \in L^2(\mathbb{R}^{2n})$

$$\begin{aligned}
U_D^{-1}(\ln \sqrt{2})\Phi(x,p) &= U_D(-\ln \sqrt{2})\Phi(x,p) \\
&= e^{n \ln \sqrt{2}}\Phi(e^{\ln \sqrt{2}}x, e^{\ln \sqrt{2}}p) \\
&= 2^{n/2}\Phi(\sqrt{2}x, \sqrt{2}p)
\end{aligned}$$

we get

$$
\begin{aligned}
U\Psi(x,p) &= U_D^{-1}(\ln\sqrt{2})\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i\xi_p\cdot p}\hat{\Psi}(\frac{\sqrt{2}}{2}(x-\xi_p),\frac{\sqrt{2}}{2}(x+\xi_p))\,d\xi_p\\
&= \frac{2^{n/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i\sqrt{2}\xi_p\cdot p}\hat{\Psi}(\frac{\sqrt{2}}{2}(\sqrt{2}x-\xi_p),\frac{\sqrt{2}}{2}(\sqrt{2}x+\xi_p))\,d\xi_p\\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i\xi_p'\cdot p}\hat{\Psi}(x-\frac{\xi_p'}{2},x+\frac{\xi_p'}{2})\,d\xi_p'\\
&= \mathcal{F}_p^{-1}\left[\hat{\Psi}(x-\xi_p/2,x+\xi_p/2)\right](x,p)
\end{aligned}
$$

where we made the change of variable $\xi_p' = \sqrt{2}\xi_p$. This proves eq.(5.8).

To prove eq.(5.9) consider eq. (5.8) for $\Psi(x,p)=\psi(x)\hat{\chi}^*(p)$. We have

$$
\hat{\Psi}(x,\xi_p)=\mathcal{F}_p[\psi(x)\hat{\chi}^*(p)](x,\xi_p)=\psi(x)\mathcal{F}_p[\hat{\chi}^*(p)](\xi_p)
$$

since

$$
\hat{\chi}^*(p)=\overline{\mathcal{F}_{\xi_p}[\chi(\xi_p)](p)}=\mathcal{F}_p^{-1}[\chi^*(\xi_p)](p)
$$

we get

$$
\hat{\Psi}(x,\xi_p)=\mathcal{F}_p[\Psi(x,p)]=\psi(x)\chi^*(\xi_p)
$$

and the result follows.   □

Finally, we consider the action of the unitary transformation on the fundamental operators

Theorem 5.5. *The operator U maps the Schrödinger phase space representation (5.2) into the Moyal representation of the Heisenberg algebra on $L^2(\mathbb{R}^{2n})$*

$$
(5.10)\qquad
\begin{aligned}
\tilde{X} &= U\hat{X}U^{-1}=\hat{X}-\frac{\hat{\tilde{\Xi}}_p}{2}=x+i\tfrac{1}{2}\partial_p\\
\tilde{P} &= U\hat{P}U^{-1}=\hat{P}-\frac{\hat{\tilde{\Xi}}_x}{2}=p+i\tfrac{1}{2}\partial_x\\
\hat{\tilde{\Xi}}_x &= U\hat{\Xi}_xU^{-1}=\hat{P}+\frac{\hat{\tilde{\Xi}}_x}{2}=p-i\tfrac{1}{2}\partial_x\\
\hat{\tilde{\Xi}}_p &= U\hat{\Xi}_pU^{-1}=\hat{X}+\frac{\hat{\tilde{\Xi}}_p}{2}=x-i\tfrac{1}{2}\partial_p
\end{aligned}
$$

Proof. We first notice that the operators $\hat{H}_D$ and $\hat{H}_R$ are self-adjoint and quadratic. Hence, the solution of the Heisenberg equations of motion

$$
i\frac{\partial}{\partial t}\hat{Z}=[\hat{Z},\hat{H}]\quad,\quad \hat{Z}=\hat{X},\hat{P},\hat{\Xi}_X,\hat{\Xi}_P
$$

coincides with the classical solution for both $\hat{H}=\hat{H}_D$ and $\hat{H}=\hat{H}_R$. The operators (5.10) are the $\hat{H}_R$-evolution (up to $t=\pi/4$) of the $\hat{H}_D$-evolution (up to $t=\ln\sqrt{2}$) of $\hat{X},\hat{P},\hat{\Xi}_X$ and $\hat{\Xi}_P$. The solution (5.10) then follows from the equivalent classical solutions that were obtained in section 1.1. We have, for instance, for $\tilde{X}$

$$
\tilde{X}=U\hat{X}U^{-1}=e^{i(\ln\sqrt{2})\hat{H}_D}e^{i\frac{\pi}{4}\hat{H}_R}\hat{X}e^{-i\frac{\pi}{4}\hat{H}_R}e^{-i(\ln\sqrt{2})\hat{H}_D}
$$

$$= e^{i(\ln\sqrt{2})\hat{H}_D} \frac{\sqrt{2}}{2} \left(\hat{X} - \hat{\bar{\Xi}}_p \right) e^{-i(\ln\sqrt{2})\hat{H}_D} = \frac{\sqrt{2}}{2} \left(\sqrt{2}\hat{X} - \frac{\sqrt{2}}{2}\hat{\bar{\Xi}}_p \right) = \hat{X} - \frac{\hat{\bar{\Xi}}_p}{2}.$$

Finally, since the transformation is unitary, the commutation relations are preserved and the operators (5.10) provide a phase space representation of the Heisenberg algebra. $\square$

5.2. **Moyal-Weyl pseudo-differential calculus.** A generic operator in the Moyal representation can be obtained from the corresponding operator in the phase space Schrödinger representation by the action of the unitary transformation U. For the operators $\hat{A}^W$ the action of U yields the *Moyal-Weyl pseudo-differential operators*

$$(5.11) \qquad \tilde{A}^W = U\hat{A}^W U^{-1} = \left(\frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} F_\sigma a(z_0) U \hat{T}_{PS}(z_0) U^{-1} \, dz_0$$

where the domain of U (5.8) was trivially extended to $\mathcal{S}'(\mathbb{R}^{2n})$.

We then have

Theorem 5.6. *The Moyal-Heisenberg-Weyl operator*

$$\tilde{T}_M(z_0) := U\hat{T}_{PS}(z_0)U^{-1} : L^2(\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}^{2n})$$

is given explicitly by ($z_0 = (x_0, \xi_{x0}) \in \mathbb{R}^{2n}$)

$$(5.12) \qquad \tilde{T}_M(x_0, \xi_{x0})\Psi(x, p) = e^{-i(x_0 \cdot p - \xi_{x0} \cdot x)} \Psi(x - \frac{x_0}{2}, p - \frac{\xi_{x0}}{2}).$$

Proof. We recall from Remark 4.4 that $\hat{T}_{PS}(z_0)$ is the unitary operator

$$\hat{T}_{PS}(z_0) = e^{-it\hat{H}_{z_0}} \Big|_{t=1}$$

associated with the infinitesimal generator

$$\hat{H}_{z_0} = \sigma(\hat{Z}_x, z_0) = x_0 \cdot \hat{\bar{\Xi}}_x - \xi_{x0} \cdot \hat{X}$$

Let now

$$\tilde{T}(z_0, t) = U e^{-it\hat{H}_{z_0}} U^{-1}$$

Then $\tilde{T}_M(z_0) = \tilde{T}(z_0, 1)$ and $\Psi(z, t) = \tilde{T}(z_0, t)\Psi_0(z)$ is the unique solution of the initial value problem

$$(5.13) \qquad i\frac{\partial}{\partial t}\Psi = U\hat{H}_{z_0}U^{-1}\Psi, \quad \Psi(\cdot, 0) = \Psi_0 \in \mathcal{S}(\mathbb{R}^{2n})$$

Since (cf. eq.(5.10))

$$U\hat{H}_{z_0}U^{-1} = x_0 \cdot \hat{P} - \xi_{x0} \cdot \hat{X} + \tfrac{1}{2}(x_0 \cdot \hat{\bar{\Xi}}_x + \xi_{x0} \cdot \hat{\bar{\Xi}}_p)$$

it is trivial to check that

$$\Psi(z, t) = e^{-i(x_0 \cdot p - \xi_{x0} \cdot x)t}\Psi_0(z - \tfrac{1}{2}z_0 t)$$

is a solution of eq.(5.13). It follows that $\tilde{T}(z_0, t)$ is unitary and extends trivially to $L^2(\mathbb{R}^{2n})$. Hence, $\tilde{T}_M(z_0) = \tilde{T}(z_0, 1)$ is, in fact, the unitary transformation given by eq.(5.12). $\square$

Theorem 5.7. *Let $a(z_x) \in \mathcal{S}(\mathbb{R}^{2n})$ be the Weyl symbol of $\hat{a}^W$. The operator $\tilde{A}^W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$ given by (5.11), explicitly*

$$(5.14) \qquad \tilde{A}^W \Psi(z) = \frac{1}{(2\pi)^n} \langle F_\sigma a(\cdot), \tilde{T}_M(\cdot)\Psi(z) \rangle$$

is a Weyl operator with symbol

$$A_M(x, p, \xi_x, \xi_p) = a(x - \frac{\xi_p}{2}, p + \frac{\xi_x}{2})$$

and we shall write a $\overset{Moyal\text{-}Weyl}{\longleftrightarrow} \tilde{A}^W$ for the "Moyal-Weyl correspondence" between a and $\tilde{A}^W$.

Proof. A simple proof follows from Theorem 4.2 and the fact that $U^{-1} \in \mathrm{Mp}(4n, \sigma_x \oplus \sigma_p)$ (because it is a unitary transformation generated by quadratic Hamiltonians) and projects into the symplectic transformation $S \in Sp(4n; \sigma_x \oplus \sigma_p)$

$$S(x, p, \xi_x, \xi_p) = (x - \frac{\xi_p}{2}, p - \frac{\xi_x}{2}, p + \frac{\xi_x}{2}, x + \frac{\xi_p}{2})$$

that was calculated explicitly in section 1.1. Hence, a direct application of Theorem 4.2 leads to the conclusion that

$$\tilde{A}^W = U\hat{A}^W U^{-1}$$

is the Weyl operator with symbol $A_M = A \circ S$ (where $A \overset{\text{Weyl}}{\longleftrightarrow} \hat{A}^W$), explicitly

$$A_M(x, p, \xi_x, \xi_p) = A(x - \frac{\xi_p}{2}, p - \frac{\xi_x}{2}, p + \frac{\xi_x}{2}, x + \frac{\xi_p}{2})$$

Since A satisfies (4.14) we also get

$$A_M(x, p, \xi_x, \xi_p) = a(x - \frac{\xi_p}{2}, p + \frac{\xi_x}{2})$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5.3. **Deformation Quantization.** In this section we succinctly discuss the relation between the Moyal representation and the deformation quantization of Bayen *et al.* [5, 6]. For a complete presentation the reader should refer to [20, 21, 18]

Theorem 5.8. *Let $\tilde{A}^W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}'(\mathbb{R}^{2n})$ be the Moyal-Weyl operator (5.14) written in terms of the Weyl symbol $a \in \mathcal{S}'(\mathbb{R}^{2n})$ of $\hat{a}^W$, i.e. a $\overset{Moyal\text{-}Weyl}{\longleftrightarrow} \tilde{A}^W$. Then, for all $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$, we have*

$$\tilde{A}^W \Psi = a \star \Psi$$

where $*$ is the Moyal starproduct (4.5).

Proof. For completeness we review the proof of [20].

For $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ we have

$$\tilde{A}^W \Psi(z) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \mathcal{F}_\sigma a(z_0) \tilde{T}_M(z_0) \Psi(z) \, dz_0$$

$$= \left(\frac{1}{2\pi}\right)^{2n} \int_{\mathbb{R}^{2n}} \left[\int_{\mathbb{R}^{2n}} e^{-i\sigma(z_0, z_1)} a(z_1) \, dz_1\right] e^{-i\sigma(z, z_0)} \Psi(z - \frac{1}{2}z_0) \, dz_0$$

Letting $z_0 = v$ and $z_1 = z + \frac{1}{2}u$ and noticing that

$$\sigma(z_0, z + \frac{1}{2}u) + \sigma(z, z_0) = \frac{1}{2}\sigma(z_0, u)$$

we find

$$\tilde{A}^W \Psi(z) = \left(\frac{1}{4\pi}\right)^{2n} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{2}\sigma(v, u)} a(z + \frac{1}{2}u) \Psi(z - \frac{1}{2}v) \, du dv$$

which is exactly $a \star \Psi$ cf.(4.5). $\qquad\square$

Two immediate corollaries are

Corollary 5.9. Ψ_λ *is the right-stargenfunction of a, i.e.*

$$a \star \Psi_\lambda = \lambda \Psi$$

iff Ψ_λ is an eigenfunction of $\tilde{A}^W \overset{Moyal\text{-}Weyl}{\longleftrightarrow} a$, i.e.

$$\tilde{A}^W \Psi_\lambda = \lambda \Psi_\lambda.$$

and:

Corollary 5.10. *The Schrödinger equation in the Moyal representation can be written in the form*

$$i\frac{\partial \Psi}{\partial t} = h \star \Psi$$

where $h \overset{Moyal\text{-}Weyl}{\longleftrightarrow} \tilde{H}^W$ and $\tilde{H}^W$ is the Hamiltonian of the system in the Moyal representation.

In view of the spectral relation of $\hat{a}^W$ and $\hat{A}^W$ (Corollary 4.5) and the unitary relation between $a\star = \tilde{A}^W = U\hat{A}^W U^{-1}$ and $\hat{A}^W$ (cf. eq.(5.11)), we also have

Corollary 5.11. *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ be the Weyl symbol of $\hat{a}^W$. Then*

i) The eigenvalues of the stargenvalue equations

(5.15) $$a \star \Psi_\lambda = \lambda \Psi_\lambda$$

and

(5.16) $$\hat{a}^W \psi_\lambda = \lambda \psi_\lambda$$

are the same.

ii) If ψ_λ is an eigenfunction (solution of eq.(5.16)) then $\Psi_\lambda = UT_\chi \psi_\lambda$ is a stargenfunction (solution of (5.15)) associated with the same eigenvalue.

iii) Conversely, if Ψ_λ is a stargenfunction and $\psi_\lambda = T_\chi^* U^{-1} \Psi_\lambda \neq 0$ then ψ_λ is an eigenfunction of $\hat{a}^W$ (associated with the same eigenvalue).

Proof. The result follows from Corollary 4.5 by a direct application of the unitary transformation U (taking into account Corollary 5.10). $\qquad\square$

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